

NONPARAMETRIC HYPERSURFACES MOVING BY POWERS OF GAUSS CURVATURE

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ABSTRACT. We study asymptotic behavior of nonparametric hypersurfaces moving by α powers of Gauss curvature with $\alpha > 1/n$. Our work generalizes the results of V. Oliker [Oli91] for $\alpha = 1$.

1. INTRODUCTION

Let Ω be a bounded strictly convex domain in \mathbb{R}^n , $n \geq 2$, with smooth boundary $\partial\Omega$. We consider a solution of the following initial boundary problem

$$(1.1) \quad \begin{aligned} u_t &= \frac{[\det(u_{ij})]^\alpha}{(1 + |\nabla u|^2)^{\alpha\beta}} \text{ in } \Omega \times (0, \infty), \\ u(x, t) &= 0 \text{ in } \partial\Omega \times (0, \infty), \\ u(x, t) &\text{ is strictly convex for each } t \geq 0, \end{aligned}$$

where $\alpha > 1/n$ and $\beta \geq 0$ are constants and

$$u_t := \frac{\partial u}{\partial t}, \quad u_{ij} := \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad \nabla u := \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right).$$

Equation (1.1) describes the graphs $(x, u(x, t))$, $(x, t) \in \bar{\Omega} \times [0, \infty)$ evolving in \mathbb{R}^{n+1} with relative boundaries $(x, u(x, t))|_{\partial\Omega}$ remain fixed. When $\beta = \frac{n+2-\frac{1}{\alpha}}{2}$, the normal speed of the point $(x, u(x, t))$ is equal to α powers of the Gauss curvature of the graph. Such parabolic Monge-Ampère equations have been studied by many authors in recent years. See, for instance, [HL06][DS12]. On the other hand, in the parametric setting, flow by Gauss curvature or its powers have received considerable interests, see [Tso85][Cho85][Cho91][And99][And00][GN][AGN] and the references therein.

V. Oliker considered (1.1) with $\alpha = 1$ in [Oli91]. He analyzed the asymptotic behavior of smooth convex solutions of (1.1). It turned out that solutions with different β all have the same asymptotic behavior. Moreover, if Ω is centrally symmetric or rotationally symmetric, then the solution $u(x, t)$ asymptotically becomes centrally symmetric or rotational symmetric, regardless of its initial shape.

The goal of this paper is to generalize V. Oliker's results in [Oli91] to any power $\alpha > 1/n$. We investigate the asymptotic behavior of a smooth convex solution of (1.1) and show

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that, by comparing with self-similar solutions of (1.1) with $\beta = 0$, the solution $u(x, t)$ asymptotically converges to the solution of the following nonlinear elliptic problem:

$$\begin{aligned} [\det(\psi_{ij})]^\alpha &= \frac{1}{1-n\alpha} \psi \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega, \\ \psi &\text{ is strictly convex and } \psi < 0 \text{ in } \Omega. \end{aligned}$$

Furthermore, our estimate implies geometric properties of the flow by α powers of the Gauss curvature. For instance, the asymptotic behavior of $u(x, t)$ reflects the symmetries of Ω . More precisely, if Ω is centrally or rotationally symmetric, then the solution $u(x, t)$ asymptotically becomes centrally or rotational symmetric, regardless of its initial shape, and we also give sharp estimates on the rate of this process.

Throughout out the paper, we denote by M the Monge-Ampère operator $M(u) := \det(u_{ij})$ and $M^\alpha(u) := [\det(u_{ij})]^\alpha$.

2. MAIN RESULTS

Consider the following initial boundary problem:

$$\begin{aligned} (2.1) \quad u_t &= M^\alpha(u) \text{ in } \Omega \times (0, \infty), \\ u(x, t) &= 0 \text{ in } \partial\Omega \times (0, \infty), \\ u(x, t) &\text{ is strictly convex for each } t \geq 0. \end{aligned}$$

We seek for self-similar solutions of (2.1) of the form

$$(2.2) \quad u(x, t) = \varphi(t)\psi(x),$$

where $\varphi(t) \in C^\infty([0, \infty))$ and $\psi(x) \in C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$. By convexity of $u(x, 0) = \varphi(0)\psi(x)$, we have either $\varphi(0) < 0$ and $\psi(x) > 0$ in Ω and concave or $\varphi(0) > 0$ and $\psi(x) < 0$ in Ω and convex. Since both cases are equivalent for our purpose, we always deal with the latter one. Substituting (2.2) into (2.1) yields

$$\frac{\varphi(t)}{\varphi^{n\alpha}} = \frac{M^\alpha(\psi)}{\psi} = \lambda = \text{constant}.$$

Noting that $\psi(x) < 0$ and convex in Ω , we get $\lambda \leq 0$ and

$$(2.3) \quad \varphi(t) = (\varphi(0)^{1-n\alpha} - (n\alpha - 1)\lambda t)^{\frac{1}{1-n\alpha}},$$

$$(2.4) \quad M(\psi) = (\lambda\psi)^{\frac{1}{\alpha}} \text{ in } \Omega \text{ and } \psi = 0 \text{ on } \partial\Omega.$$

An easy argument shows that $\lambda = 0$ implies $u(x, t) \equiv 0$. Thus we only consider the case $\lambda < 0$. By scaling, it suffices to consider one negative value of λ and thus we fix $\lambda = \frac{1}{1-n\alpha} < 0$ for convenience. The following result establishes the existence of self-similar solutions to (2.1).

Theorem 2.1. *Let Ω be a bounded strictly convex domain with smooth boundary $\partial\Omega$. Then problem (2.1) admits a self-similar solution in $\overline{\Omega} \times (0, \infty)$ given by*

$$(2.5) \quad u(x, t) = (1 + t)^{\frac{1}{1-n\alpha}} \psi(x),$$

where ψ is the unique solution in $C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ of the equation

$$(2.6) \quad M(\psi) = \left(\frac{-\psi}{|1 - n\alpha|} \right)^{\frac{1}{\alpha}} \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega,$$

ψ is strictly convex and $\psi < 0$ in Ω ,

and $\sup_\Omega |\psi(x)|$ admits an estimate depending only on n, α and the domain Ω . Furthermore, if $\tilde{u}(x, t) = \varphi(t)\tilde{\psi}(x)$ is an arbitrary self-similar solution of (2.1), then there exists a unique $c > 0$ such that $\tilde{\psi}(x) = c\psi(x)$ and

$$(2.7) \quad \tilde{u}(x, t) = u(x, t) \left\{ \frac{1+t}{[c\varphi(0)]^{1-n\alpha} + t} \right\}^{\frac{1}{n\alpha-1}}.$$

The main theorem concerning the asymptotic behavior of the solution is the following:

Theorem 2.2. *Let $u(x, t) \in C^2(\overline{\Omega} \times (0, \infty))$ be a solution of the problem*

$$(2.8) \quad \begin{aligned} u_t &= \frac{M^\alpha(u)}{(1 + |\nabla u|^2)^{\alpha\beta}} \quad \text{in } \Omega \times (0, \infty), \\ u(x, t) &= 0 \quad \text{in } \partial\Omega \times (0, \infty), \\ u(x, t) &\text{ is strictly convex for each } t \geq 0, \end{aligned}$$

where $\alpha > 1/n$ and $\beta \geq 0$ are constants. If $\beta = 0$, then there exists positive constant C_1 depending only on dimension n, α, Ω and $u(x, 0)$, such that for all $t \geq 0$,

$$(2.9) \quad \sup_\Omega \left| (1+t)^{\frac{1}{n\alpha-1}} u(x, t) - \psi(x) \right| \leq \frac{C_1}{1+t},$$

If $\beta > 0$, then

$$(2.10) \quad \left[\frac{C_2}{1+t} + G^{\frac{1}{1-n\alpha}} - 1 \right] \psi \leq (1+t)^{\frac{1}{n\alpha-1}} u(x, t) - \psi(x) \leq \frac{-C_3\psi}{1+t},$$

where C_2 and C_3 are positive constants depending only on dimension $n, \alpha, \Omega, u(x, 0)$ and

$$G = \inf_\Omega (1 + |\nabla u(x, 0)|^2)^{-\alpha\beta}.$$

Moreover,

$$(2.11) \quad \lim_{t \rightarrow \infty} (1+t)^{\frac{1}{n\alpha-1}} u(x, t) = \psi(x) \quad \text{uniformly on } \overline{\Omega}.$$

We have gradient estimates for solutions of (2.8).

Corollary 2.3. *Suppose the same conditions as in Theorem 2.2 holds. Then for all $t \geq 0$,*

$$\sup_\Omega |\nabla u(x, t)| \leq G^{\frac{1}{1-n\alpha}} \sup_{\partial\Omega} \psi_\nu(x) (C_4 + t)^{\frac{1}{1-n\alpha}}$$

where ψ_ν is the derivative in the direction of the outward unit normal to $\partial\Omega$, and C_4 depends only on $u(x, 0)$.

An interesting geometric consequence of Theorem 2.2 is the following:

Theorem 2.4. *If Ω is a ball in \mathbb{R}^n and $u(x, t) \in C^2(\overline{\Omega} \times (0, \infty))$ is a solution of (2.8). Then*

$$(1+t)^{\frac{1}{n\alpha-1}} u(x, t) \rightarrow \psi(|x|) \quad \text{uniformly on } \overline{\Omega} \text{ as } t \rightarrow \infty.$$

This theorem implies that, $u(x, t)$ asymptotically becomes radially symmetric regardless of the initial shape. More generally, if Ω is centrally symmetric, then

$$(1+t)^{\frac{1}{n\alpha-1}}u(x, t) \rightarrow \psi(x) \text{ uniformly on } \overline{\Omega} \text{ as } t \rightarrow \infty,$$

where $\psi(x) = \psi(-x)$. The proof of Theorem 2.4 is the same as in [Oli91, Section 6] and we omit it here.

3. PROOF OF THEOREM 2.1

Proof. It was shown in [Tso90, Corollary 4.2, in which (2) should read as (1.2)] that for any $\alpha > 1/n$, problem (2.6) admits a unique strictly convex solution ψ in $C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$. Direct calculation shows $u(x, t) = (1+t)^{\frac{1}{1-n\alpha}}\psi(x)$ solves (2.1) with initial data $u_0(x) = \psi(x)$. Next we prove $\sup_\Omega |\psi(x)|$ depends only on n, α and Ω . Since ψ is strictly convex and vanishes on $\partial\Omega$, there exists a point $\bar{x} \in \Omega$ such that $\sup_\Omega |\psi| = |\psi(\bar{x})|$. Consider a cone K generated by the linear segments joining the vertex $(\bar{x}, \psi(\bar{x}))$ with points on $\partial\Omega$. Denote $\theta(x), x \in \overline{\Omega}$, the function whose graph is K . Obviously, $\theta \geq \psi$ in Ω and $\theta = \psi = 0$ on $\partial\Omega$. Then by [Gut01, Lemma 1.4.1] $M\theta(\Omega) \leq M\psi(\Omega)$, where Mu denotes the Monge-Ampère measure associated with the function u (see [Gut01, Theorem 1.1.13]). Since ψ is C^∞ and convex on Ω ,

$$(3.1) \quad M\psi(\Omega) = \int_\Omega M(\psi) = \int_\Omega (\lambda\psi)^{\frac{1}{\alpha}} \leq |\lambda|^{\frac{1}{\alpha}} |\psi(\bar{x})|^{\frac{1}{\alpha}} |\Omega|.$$

On the other hand, the Aleksandrov-Bakelman-Pucci maximum principle (see, for instance, [Gut01, Theorem 1.4.5]) says $M\theta(\Omega) \geq \omega_n |\psi(\bar{x})|^n (\text{diam}\Omega)^{-n}$, where ω_n is the volume of the unit ball in \mathbb{R}^n . Thus

$$(3.2) \quad \sup_\Omega |\psi(x)| = |\psi(\bar{x})| \leq \left(\frac{|\lambda|^{\frac{1}{\alpha}} |\Omega| (\text{diam}\Omega)^n}{\omega_n} \right)^{\frac{\alpha}{n\alpha-1}}.$$

Finally, the proof of (2.7) parallels that in [Oli91, Section 4.3]. \square

Remark 3.1. One can prove Theorem 2.1 without using the existence results from [Tso90]. V. Oliker [Oli91] proved that (2.6) has a unique solution in $C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ when $\alpha = 1$. A careful examination of his proof shows it works indeed for all $\alpha > 1/n$.

Remark 3.2. When $\alpha = 1/n$, it was shown by P. L. Lions [Lio83] that

$$(3.3) \quad M(\psi) = \mu(-\psi)^n \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega$$

admits a unique solution pair (μ, ψ) in the sense that if (ν, ϕ) , where ν is positive and ϕ is convex, solves (3.3), then we must have $\mu = \nu$ and ϕ is a constant multiple of ψ . The number μ is called the first (in fact the only) eigenvalue of the Monge-Ampère operator M , and the corresponding (normalized) eigenfunction is in $C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$. The asymptotic behavior for $\alpha = 1/n$ remains interesting and open.

Remark 3.3. When $0 < \alpha < 1/n$, K. Tso [Tso90, Theorem E] showed that (2.6) admits a convex solution in $C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$. The uniqueness, however, is not known. In this case, the reader will see easily from the comparison with self-similar supersolutions in Section 4 that smooth convex solutions of (2.8) must vanish at finite time.

4. PROOF OF THEOREM 2.2

In this section, we determine the asymptotic behavior of u by comparing with self-similar solutions of (2.1). A direct generalization of the proof given by V. Olier in [Oli91] works for $\alpha \geq 2/n$. New estimates are introduced in the following lemma to take care of the case $1/n < \alpha < 2/n$.

Lemma 4.1. *Let $F : (0, S) \times [0, \infty) \rightarrow (0, \infty)$, $S < \infty$ be defined by*

$$(4.1) \quad F(s, t) = \left(\frac{1+t}{s+t} \right)^{\frac{1}{n\alpha-1}} \equiv \left(1 + \frac{1-s}{s+t} \right)^{\frac{1}{n\alpha-1}}.$$

Then we have for all $t \geq 0$,

$$(4.2) \quad F(s, t) \leq 1 + \frac{1}{n\alpha-1} \frac{1-s}{s(1+t)}, \quad \text{if } s \leq 1, \alpha \geq 2/n;$$

$$(4.3) \quad F(s, t) \leq 1 + \frac{1}{n\alpha-1} \left(\frac{1}{s} \right)^{\frac{1}{n\alpha-1}} \frac{1-s}{1+t}, \quad \text{if } s \leq 1, \alpha \leq 2/n;$$

$$(4.4) \quad F(s, t) \geq 1 - \frac{s-1}{1+t}, \quad \text{if } s \geq 1, \alpha \geq 2/n;$$

$$(4.5) \quad F(s, t) \geq 1 - \frac{1}{n\alpha-1} \frac{s-1}{1+t} \quad \text{if } s \geq 1, \alpha \leq 2/n.$$

Proof. This lemma follows from elementary calculus. When $\alpha \geq 2/n$, $\gamma := \frac{1}{n\alpha-1} \leq 1$. Then (4.2) follows from $(1+x)^\gamma \leq 1 + \gamma x$ for all $x \geq 0$ and (4.4) follows from $x^\gamma \geq x$ for all $0 \leq x \leq 1$. When $\alpha \leq 2/n$, $\gamma := \frac{1}{n\alpha-1} \geq 1$. Now (4.3) is a consequence of $(1+x)^\gamma \leq 1 + \gamma(1+a)^{\gamma-1}x$ for all $0 \leq x \leq a$ and (4.5) is a consequence of $(1+x)^\gamma \geq 1 + \gamma x$ for all $-1 < x \leq 0$. \square

Proof of Theorem 2.2. First of all, a uniform estimate of $|\nabla u(x, t)|$ is obtained similarly as in [Oli91]. For any $t \geq 0$,

$$(4.6) \quad \sup_{\bar{\Omega}} |\nabla u(x, t)| \leq \sup_{\partial\Omega} |\nabla u(x, t)| = \sup_{\partial\Omega} |u_\nu(x, t)| \leq \sup_{\partial\Omega} |u_\nu(x, 0)|.$$

Self-similar subsolution and supersolution are then constructed as follows: Let

$$G = \inf_{\Omega} (1 + |\nabla u(x, 0)|^2)^{-\alpha\beta}.$$

Clearly we have $0 < G \leq 1$. It follows from (4.6) that

$$(4.7) \quad GM^\alpha(u) \leq (1 + |\nabla u(x, t)|^2)^{-\alpha\beta} M^\alpha(u) = u_t \text{ in } \Omega \times (0, \infty).$$

Put $\underline{u}(x, t) = G^{\frac{1}{1-n\alpha}} \underline{\varphi}(t) \psi(x)$ and $\bar{u}(x, t) = \bar{\varphi}(t) \psi(x)$, where ψ is the solution of (2.6) and

$$\begin{aligned} \underline{\varphi}(t) &= (\underline{\varphi}(0)^{1-n\alpha} + t)^{\frac{1}{1-n\alpha}}, \\ \bar{\varphi}(t) &= (\bar{\varphi}(0)^{1-n\alpha} + t)^{\frac{1}{1-n\alpha}}. \end{aligned}$$

Then \underline{u} and \bar{u} satisfy $\underline{u}_t = GM^\alpha(\underline{u})$ and $\bar{u}_t = M^\alpha(\bar{u})$ in $\Omega \times (0, \infty)$, respectively. Finally we define $\tilde{u}(x, t) = \underline{u}(x, t) - u(x, t)$ and it satisfies

$$(4.8) \quad \tilde{u}_t = GM^\alpha(\underline{u}) - (1 + |\nabla u(x, t)|^2)^{-\alpha\beta} M^\alpha(u) \leq GM^\alpha(\underline{u}) - GM^\alpha(u) \text{ in } \Omega \times (0, \infty).$$

Observe that the operator $L(\tilde{u}) = M^\alpha(\underline{u}) - M^\alpha(u)$ is elliptic since

$$L(\tilde{u}) = \sum_{ij} \left(\int_0^1 \alpha \det(u_{\tau ij})^{\alpha-1} \text{cof}(u_{\tau ij}) d\tau \right) \tilde{u}_{ij},$$

where $u_\tau(x, t) = \tau \underline{u}(x, t) + (1 - \tau)u(x, t)$ is strictly convex and the cofactor matrix $\text{cof}(u_{\tau ij})$ is positive definite on any compact subset of $\Omega \times (0, T]$ for any $T < \infty$. Next we choose $\underline{\varphi}(0)$ and $\bar{\varphi}(0)$ so that $\underline{\varphi}(0)\psi(x) \leq u(x, 0) \leq \bar{\varphi}(0)\psi(x)$ on Ω . Then

$$(4.9) \quad \tilde{u}(x, 0) \leq 0 \text{ in } \bar{\Omega} \text{ and } \tilde{u}(x, t) = 0 \text{ in } \partial\Omega \times [0, \infty),$$

and we can then apply the classical maximum principle to conclude that $\tilde{u}(x, t) = \underline{u}(x, t) - u(x, t) \leq 0$ on $\bar{\Omega} \times [0, \infty)$. Consequently,

$$(4.10) \quad \left\{ (1+t)^{\frac{1}{n\alpha-1}} (G(\underline{\varphi}(0)^{1-n\alpha} + t))^{\frac{1}{1-n\alpha}} - 1 \right\} \psi(x) \leq (1+t)^{\frac{1}{n\alpha-1}} u(x, t) - \psi(x).$$

Similarly, one derives that $u(x, t) \leq \bar{u}(x, t)$, namely,

$$(4.11) \quad (1+t)^{\frac{1}{n\alpha-1}} u(x, t) - \psi(x) \leq \left\{ (1+t)^{\frac{1}{n\alpha-1}} (\bar{\varphi}(0)^{1-n\alpha} + t)^{\frac{1}{1-n\alpha}} - 1 \right\} \psi(x)$$

Without loss of generality we may assume $\underline{\varphi}(0) \geq 1$ and $\bar{\varphi}(0) \leq 1$. Thus by Lemma 4.1,

$$F(\underline{\varphi}(0)^{1-n\alpha}, t) \leq 1 + C_2/(1+t)$$

$$F(\bar{\varphi}(0)^{1-n\alpha}, t) \geq 1 - C_3/(1+t),$$

where C_2, C_3 depend on n, α and $u_0(x)$. Combining now (4.10) and (4.11), we arrive at that for all $t \geq 0$ and $x \in \bar{\Omega}$,

$$(4.12) \quad \left[\frac{C_2}{1+t} + G^{\frac{1}{1-n\alpha}} - 1 \right] \psi \leq (1+t)^{\frac{1}{n\alpha-1}} u(x, t) - \psi \leq \frac{-C_3\psi}{1+t},$$

If $\beta = 0$, then $G = 1$ and (4.12) implies (2.9) with $C_1 = \max\{C_2, C_3\} \sup_\Omega |\psi|$. If $\beta > 0$, one needs to estimate $|\nabla u(x, t)|$ more carefully as V. Oliker did [Oli91, Pages 255-256]. Take an increasing sequence $t_m \rightarrow \infty$ and let $G_m = \inf_\Omega (1 + |\nabla u(x, t_m)|^2)^{-\alpha\beta}$. The same argument as in deriving (4.12) yields for all $t \geq t_m$ and $x \in \bar{\Omega}$,

$$(4.13) \quad \left[\frac{c_m}{1+t} + G_m^{\frac{-1}{n\alpha-1}} - 1 \right] \psi \leq (1+t)^{\frac{1}{n\alpha-1}} u(x, t) - \psi \leq \frac{-C_3\psi}{1+t}.$$

where $c_m = (1 - \underline{\varphi}(t_m)^{1-n\alpha}) \underline{\varphi}(t_m)^{n\alpha} / (n\alpha - 1) < \infty$ uniformly in m due to (4.10). The same argument as in [Oli91] allows one to let $t_m \rightarrow \infty$ and deduce (2.11), hence completing the proof of Theorem 2.2. \square

Remark 4.1. Similarly to [AP81] one sees the sharpness of the estimate (4.13) by considering the function $u(x, t) = (s + t)^{\frac{1}{n\alpha-1}} \psi(x)$ for any $s > 0$.

Remark 4.2. Corollary 2.3 with $C_4 = \underline{\varphi}(0)^{1-n\alpha}$ follows from $\underline{u}(x, t) \leq u(x, t)$, namely,

$$G^{\frac{1}{1-n\alpha}} (\underline{\varphi}(0)^{1-n\alpha} + t)^{\frac{1}{1-n\alpha}} \psi(x) \leq u(x, t).$$

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